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Einstein metrics with A_{16}-type holonomy

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Abstract
Einstein neutral metrics in dimension four are constructed. They provide examples of holonomy type A_{16}.

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The holonomy group of a metric g at a point x of a manifold M is the group of linear transformations in the tangent space of x defined by parallel translation along all possible loops starting at x. For connections on connected manifolds, holonomy groups of different points are isomorphic, and so we shall refer to *the* holonomy group of g and denote it by $Hol(g)$ [1].

If one restricts to curves which are null homotopic, one obtains the restricted holonomy group $Hol^{\circ}(g)$ of M, which is the identity component of $Hol(g)$. Clearly $Hol(g)$ and $Hol^{\circ}(g)$ are equal if M is simply connected. Also, $Hol(g)$ and $Hol^{\circ}(g)$ are Lie groups.

It is obvious that a connection can only be the Levi-Civita connection of a metric g if the holonomy group is a subgroup of the generalized orthogonal group corresponding to the signature of g [2].

At any point $x \in M$, and in some coordinate system about x, the set of matrices of the form

$$\mathbf{R^a_{bcd}X^cY^d} \qquad \mathbf{R^a_{bcd;e}X^cY^dZ^e} \qquad \mathbf{R^a_{bcd;ef}X^cY^dZ^eW^f} \cdots$$

where $X, Y, Z, W \in T_xM$ and a semi-colon denotes a covariant derivative, forms a Lie subalgebra of the Lie algebra of $M_n(\mathbb{R})$ of $GL(n,\mathbb{R})$ called the *infinitesimal holonomy* algebra of M at x. Up to isomorphism the latter is independent of the coordinate system chosen and is denoted by $hol'(g)$. The corresponding uniquely determined connected subgroup of $GL(n,\mathbb{R})$ is called the infinitesimal holonomy group of M at x and is denoted by $Hol'(g)$. The Lie algebra $hol'(g)$ is a subalgebra of the Lie algebra of $Hol(g)$ for each $x \in M$ [3–5].

In a recent paper, Ghanam and Thompson [6] studied and classified the holonomy Lie subalgebras of neutral metrics in dimension four. In this paper, we will study one of these subalgebras A_{16} and construct Einstein metrics of holonomy type A_{16}.

A_{16} is a two-dimensional Lie subalgebra generated by $\left[\begin{smallmatrix} -J & 0 \\ 0 & J \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right]$ where $J = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. By a change of basis, one obtains a new set of generators, namely, $\left[\begin{smallmatrix} J & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & 0 \\ 0 & J \end{smallmatrix}\right]$. Consider $\mathbb{R}^2$ with coordinates (x, y), and let

$$g = \begin{bmatrix} a & c \\ c & b \end{bmatrix} \tag{1}$$

be a two-dimensional metric, where a, b and c are smooth functions of x and y. Then the equation $gJ + (gJ)^t = 0$ implies that $a = b$ and $c = 0$. Hence $g = a(x, y)I$, where I is the 2×2 identity matrix. We apply the de Rham decomposition theorem: use a pair of non-flat two-dimensional Riemannian metrics with the signs adjusted so as to produce a neutral four-dimensional metric. This proves that A_{16} is a holonomy Lie algebra of four-dimensional neutral metric.

Let us turn to equation (1). The non-zero components of the Ricci tensor are

$$R_{11} = R_{22} = \frac{\left(a_{yy}a + a_{xx}a - a_y^2 - a_x^2\right)}{2a^2} \tag{2}$$

and the Ricci scalar is given by

$$R = \frac{\left(a_{yy}a + a_{xx}a - a_y^2 - a_x^2\right)}{a^3}. \tag{3}$$

The Einstein condition entails that

$$R_{ij} = \tfrac{1}{4} R g_{ij} \tag{4}$$

and so if we take $i = j = 1$ in equation (4), we obtain the following PDE:

$$a(a_{xx} + a_{yy}) - \left(a_x^2 + a_y^2\right) = 0. \tag{5}$$

We will solve equation (5) by separation of variables. Let

$$a(x, y) = f(x)h(y) \tag{6}$$

where f and h are smooth functions of x and y, respectively. Equation (5) becomes

$$fh(f''h + fh'') - ((f')^2h^2 + f^2(h')^2) = 0 \tag{7}$$

and so

$$h^2(ff'' - (f')^2) + f^2(hh'' - (h')^2) = 0. \tag{8}$$

If we divide equation (8) by f^2h^2 we obtain

$$\frac{ff'' - (f')^2}{f^2} = \frac{(h')^2 - hh''}{h^2}. \tag{9}$$

This implies that

$$\frac{ff'' - (f')^2}{f^2} = \frac{(h')^2 - hh''}{h^2} = c \tag{10}$$

where c is a constant. Now we will consider two cases:

Case (1). If $c = 0$, then equation (10) gives

$$ff'' - (f')^2 = 0 \tag{11}$$

and

$$(h')^2 - hh'' = 0. \tag{12}$$

To solve equation (11) we use the following substitution:

$$f(x) = \mathrm{e}^{z(x)} \tag{13}$$

where $z(x)$ is a smooth function of x. We substitute equation (13) in equation (11) to obtain

$$\mathrm{e}^z(z''\mathrm{e}^z + (z')^2\mathrm{e}^z) - (z')^2\mathrm{e}^{2z} = 0 \tag{14}$$

and so

$$z'' = 0. \tag{15}$$

Hence

$$z(x) = c_1x + c_2 \tag{16}$$

where c_1 and c_2 are constants. Hence

$$f(x) = \mathrm{e}^{c_1x+c_2}. \tag{17}$$

Similarly equation (12) implies that

$$h(y) = \mathrm{e}^{c_3y+c_4} \tag{18}$$

where c_3 and c_4 are constants. Therefore the two-dimensional Einstein metric g is given by

$$g = \mathrm{e}^{c_1x+c_2y+c_3}(\mathrm{d}x^2 + \mathrm{d}y^2) \tag{19}$$

and so the four-dimensional Einstein neutral metric with holonomy type A_{16} is

$$g = \mathrm{e}^{c_1x+c_2y+c_3}(\mathrm{d}x^2 + \mathrm{d}y^2) - \mathrm{e}^{c_4z+c_5t+c_6}(\mathrm{d}z^2 + \mathrm{d}t^2) \tag{20}$$

where (x, y, z, t) is a coordinate system on $\mathbb{R}^4$.

Case (2). If $c \neq 0$, then equation (10) gives

$$ff'' - (f')^2 - cf^2 = 0 \tag{21}$$

and

$$(h')^2 - hh'' - ch^2 = 0. \tag{22}$$

To solve equation (21), we use equation (13) to obtain

$$z'' - c = 0. \tag{23}$$

and so

$$z(x) = \tfrac{1}{2}cx^2 + c_1x + c_2. \tag{24}$$

To solve equation (22), we let $h(y) = \mathrm{e}^{w(y)}$ to obtain

$$w'' + c = 0 \tag{25}$$

and so

$$w(y) = -\tfrac{1}{2}cy^2 + c_3y + c_4. \tag{26}$$

Thus, the two-dimensional Einstein metric g is given by

$$g = \mathrm{e}^{\frac{1}{2}c(x^2-y^2)+c_1x+c_2y+c_3}(\mathrm{d}x^2 + \mathrm{d}y^2) \tag{27}$$

and so the four-dimensional Einstein neutral metric with holonomy type A_{16} is

$$g = \mathrm{e}^{\frac{1}{2}c(x^2-y^2)+c_1x+c_2y+c_3}(\mathrm{d}x^2 + \mathrm{d}y^2) - \mathrm{e}^{\frac{1}{2}r(z^2-t^2)+c_4z+c_5t+c_6}(\mathrm{d}z^2 + \mathrm{d}t^2). \tag{28}$$

where c_1, c_2, c_3, c_4, c_5, c_6 and r are constants.

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